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Three-Dimensional Time Harmonic Electromagnetic Inverse Scattering: The Reconstruction of the Shape and the Impedance of an Obstacle

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Abstract—A numerical method for the reconstruction of the shape and the impedance of an obstacle from time harmonic electromagnetic scattering data is presented. Let D be a bounded, simply connected domain contained in the three-dimensional Euclidean space $\mathbb{R}^3$, with smooth boundary ∂D. The three-dimensional Euclidean space is filled with an isotropic homogeneous medium. We assume that D contains the origin, and D is regarded as an obstacle whose electric properties are given by a boundary impedance $\chi(\mathbf{x})$, $\mathbf{x} \in \partial D$. From the knowledge of the electric far fields generated by the obstacle D when hit by known time harmonic electromagnetic waves, the shape ∂D, and the boundary impedance $\chi(\mathbf{x})$ of the obstacle are reconstructed. The reconstruction algorithm is based on the "Herglotz function method" introduced by Colton and Monk [1] in acoustic scattering.

Keywords—Electromagnetic scattering, Inverse obstacle shape, Impedance reconstruction.

1. INTRODUCTION

Let $\mathbb{R}^3$ be the three-dimensional real Euclidean space $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ be a generic vector where the superscript $^\top$ means transposed, $(\cdot,\cdot)$ denotes the Euclidean scalar product, $\|\cdot\|$ denotes the Euclidean norm, and $\cdot \times \cdot$ denotes the Euclidean vector product. Occasionally, we abuse the notation doing the real Euclidean scalar product $(\cdot,\cdot)$ of complex vectors.

Let $D \subset \mathbb{R}^3$ be a bounded simply-connected domain with smooth boundary ∂D; we assume that D contains the origin.

We assume that all the electromagnetic fields are time harmonic. Let $\mathbf{E}^i$ be the part depending on the space variables $\mathbf{x}$ of the electric field associated to a linearly polarized incoming plane wave, that is

$$\mathbf{E}^i(\mathbf{x}) = \boldsymbol{\omega} e^{ik(\mathbf{x},\boldsymbol{\alpha})}, \tag{1.1}$$

where $\boldsymbol{\omega}, \boldsymbol{\alpha} \in \mathbb{R}^3$, $\|\alpha\| = 1$ are given and $k > 0$ is the wave number. The vector $\boldsymbol{\alpha}$ is the propagation direction of the electric plane wave and $\boldsymbol{\omega}$ is the polarization vector. We assume

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$(\boldsymbol{\omega}, \boldsymbol{\alpha}) = 0$, that is

$$\text{div } \mathbf{E}^i(\mathbf{x}) = 0, \tag{1.2}$$

where "div" denotes the divergence operator. The magnetic field associated to this incoming wave is

$$\mathbf{H}^i(\mathbf{x}) = \frac{1}{ik} \text{ curl } \mathbf{E}^i(\mathbf{x}), \tag{1.3}$$

where "curl" denotes the curl operator.

Let $\mathbf{E}^s$ be the part depending on the space variables of the electric field scattered by the obstacle D when hit by the incoming wave $\mathbf{E}^i$; we denote with

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}^i(\mathbf{x}) + \mathbf{E}^s(\mathbf{x}) \tag{1.4}$$

the total electric field. It is easy to prove, see [2], that if the region $\mathbb{R}^3 \setminus D$ is filled with a homogeneous isotropic medium that does not contain charges, then the time harmonic Maxwell's equations for the scattered time harmonic electromagnetic field can be reduced to the vector Helmholtz equation with the divergence free condition, that is

$$\Delta \mathbf{E}^s(\mathbf{x}) + k^2 \mathbf{E}^s(\mathbf{x}) = \mathbf{0}, \qquad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{1.5}$$

$$\text{div } \mathbf{E}^s(\mathbf{x}) = 0, \qquad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{1.6}$$

where Δ denotes the vector Laplace operator.

The partial differential equations (1.5),(1.6) must be equipped with boundary conditions. The first boundary condition is at infinity and is strictly related to the character of the electromagnetic fields; it is known as Silver-Müller radiation condition [2], that is

$$\text{curl } \mathbf{E}^s(\mathbf{x}) \times \hat{\mathbf{x}} - ik\mathbf{E}^s(\mathbf{x}) = o\left(\frac{1}{\|\mathbf{x}\|}\right), \qquad \|\mathbf{x}\| \to \infty, \tag{1.7}$$

where $\hat{\mathbf{x}} = (\mathbf{x}/\|\mathbf{x}\|)$, $\mathbf{x} \neq \mathbf{0}$. The second boundary condition depends on the electric properties of the obstacle; that is, let $\hat{\boldsymbol{\nu}}(\mathbf{x})$ be the exterior unit normal to ∂D in the point $\mathbf{x} \in \partial D$ and let $\chi(\mathbf{x})$, $x \in \partial D$, be a complex valued function. We assume

$$\hat{\boldsymbol{\nu}}(\mathbf{x}) \times \text{curl } \mathbf{E}(\mathbf{x}) + ik\chi(\mathbf{x})\hat{\boldsymbol{\nu}}(\mathbf{x}) \times (\hat{\boldsymbol{\nu}}(\mathbf{x}) \times \mathbf{E}(\mathbf{x})) = \mathbf{0}, \qquad \mathbf{x} \in \partial D. \tag{1.8}$$

The function χ is the boundary electric impedance of the obstacle D.

The scattered field $\mathbf{E}^s(\mathbf{x})$, unique solution of (1.5)–(1.8), see [2], has the following expansion:

$$\mathbf{E}^s(\mathbf{x}) = \frac{e^{ik\|\mathbf{x}\|}}{\|\mathbf{x}\|} \mathbf{E}_0(\hat{\mathbf{x}}, k, \boldsymbol{\alpha}, \boldsymbol{\omega}) + O\left(\frac{1}{\|\mathbf{x}\|^2}\right), \qquad \|\mathbf{x}\| \to \infty, \tag{1.9}$$

where $\mathbf{E}_0$ is the electric far field pattern generated by the interaction of the obstacle D with the incoming wave (1.1).

In this paper, we give a numerical method to recover the boundary ∂D and the boundary electric impedance χ of the obstacle D from the knowledge of the electric far field patterns generated by several incoming waves.

The algorithm is based on the "Herglotz function method" introduced in [1] in the context of inverse acoustic scattering. This algorithm is effective in the resonance region, that is, when

$$kL \approx 1, \tag{1.10}$$

where L is a characteristic length of the obstacle D.

Let λ_n, $n = 1, 2, \ldots$ be the eigenvalues of the vector Laplace operator restricted to the divergence free vector fields in the interior of D with the boundary condition (1.8). Let $B = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| < 1\}$ and ∂B be the boundary of B. We consider the following problem.

PROBLEM 1.1. *Let $\Omega_1 \subseteq \partial B$, $\Omega_2 \subseteq \partial B \times \mathbb{R}^3$, $\Omega_3 \subseteq \mathbb{R}$ be three given sets such that $-\lambda_n \notin \Omega_3$, $n = 1, 2, \ldots$. From the knowledge of $\mathbf{E}_0(\hat{\mathbf{x}}, k, \boldsymbol{\alpha}, \boldsymbol{\omega}_{\boldsymbol{\alpha}})$, for $\hat{\mathbf{x}} \in \Omega_1$, $(\boldsymbol{\alpha}, \boldsymbol{\omega}_{\boldsymbol{\alpha}}) \in \Omega_2$, $k^2 \in \Omega_3$, determine the boundary ∂D and the impedance χ of the obstacle D.*

In [3], a similar problem is considered for time harmonic acoustic scattering. We note that the condition $-\lambda_n \notin \Omega_3$, $n = 1, 2, \ldots$ is a nonresonant condition.

In Section 2, we give some mathematical relations used to solve Problem 1.1. In Section 3, we explain the numerical method used to solve Problem 1.1. In Section 4, we give some numerical results.

2. THE MATHEMATICAL FORMULATION OF THE INVERSE PROBLEM

Let $\mathbf{g}(\hat{\mathbf{x}})$ be a square integrable complex valued vector function defined on ∂B, such that

$$(\hat{\mathbf{x}}, \mathbf{g}(\hat{\mathbf{x}})) = 0, \qquad \forall \hat{\mathbf{x}} \in \partial B, \tag{2.1}$$

and let

$$\mathcal{E}(\mathbf{y}) = \int_{\partial B} \mathbf{g}(\hat{\mathbf{x}}) e^{ik(\hat{\mathbf{x}},\mathbf{y})}\, d\lambda(\hat{\mathbf{x}}), \tag{2.2}$$

where $d\lambda$ is the surface measure on ∂B. The vector field $\mathcal{E}$ satisfies the vector Helmholtz equation in $\mathbb{R}^3$; moreover, from (2.1), we have that $\mathcal{E}$ is a divergence-free vector field.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \neq \mathbf{y}$, let

$$\phi(\mathbf{x}, \mathbf{y}) = \frac{e^{ik\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|} \tag{2.3}$$

be the Green's function of the (scalar) Helmholtz operator with the Sommerfeld radiation condition; see [2]. Let $\mathbf{v}, \mathbf{a} \in \mathbb{R}^3$ be two given vectors; we define the following vector fields:

$$\mathbf{M}_1(\mathbf{y}) = -4\pi \left\{ \mathbf{v}\overline{\phi}(\mathbf{x}, \mathbf{y}) + \frac{1}{k^2} \nabla_{\mathbf{x}} \left(\mathbf{v}, \nabla_{\mathbf{x}} \overline{\phi}(\mathbf{x}, \mathbf{y}) \right) \right\} \Big|_{\mathbf{x}=\mathbf{0}}, \qquad \mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}, \tag{2.4}$$

$$\mathbf{M}_2(\mathbf{y}) = -4\pi \left\{ (\mathbf{a}, \nabla_{\mathbf{x}}) \left[\mathbf{v}\overline{\phi}(\mathbf{x}, \mathbf{y}) + \frac{1}{k^2} \nabla_{\mathbf{x}} \left(\mathbf{v}, \nabla_{\mathbf{x}} \overline{\phi}(\mathbf{x}, \mathbf{y}) \right) \right] \right\} \Big|_{\mathbf{x}=\mathbf{0}}, \qquad \mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}, \tag{2.5}$$

where $\nabla_{\mathbf{x}}$ is the gradient operator with respect to $\mathbf{x}$, and $\overline{\phi}$ is the complex conjugate of ϕ. Given a vector field $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$, let

$$(\mathbf{a}, \nabla_{\mathbf{x}})\mathbf{F}(\mathbf{x}) = ((\mathbf{a}, \nabla_{\mathbf{x}} F_1(\mathbf{x})), (\mathbf{a}, \nabla_{\mathbf{x}} F_2(\mathbf{x})), (\mathbf{a}, \nabla_{\mathbf{x}} F_3(\mathbf{x}))). \tag{2.6}$$

Let $k^2 \neq -\lambda_n$, $n = 1, 2, \ldots$, and let $\mathcal{E}_1$ be the unique solution of the boundary value problem

$$\Delta \mathcal{E}_1(\mathbf{y}) + k^2 \mathcal{E}_1(\mathbf{y}) = \mathbf{0}, \qquad \mathbf{y} \in D, \tag{2.7}$$

$$\operatorname{div} \mathcal{E}_1(\mathbf{y}) = 0, \qquad \mathbf{y} \in D, \tag{2.8}$$

$$\begin{aligned} &\operatorname{curl} \mathcal{E}_1(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) + ik\overline{\chi}(\mathbf{y}) \left((\mathcal{E}_1(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) \right) \\ &\quad = \operatorname{curl} \mathbf{M}_1(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) + ik\overline{\chi}(\mathbf{y})((\mathbf{M}_1(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})), \qquad \mathbf{y} \in \partial D; \end{aligned} \tag{2.9}$$

moreover, let $\mathcal{E}_2$ be the unique solution of the boundary value problem

$$\Delta \mathcal{E}_2(\mathbf{y}) + k^2 \mathcal{E}_2(\mathbf{y}) = \mathbf{0}, \qquad \mathbf{y} \in D, \tag{2.10}$$

$$\operatorname{div} \mathcal{E}_2(\mathbf{y}) = 0, \qquad \mathbf{y} \in D, \tag{2.11}$$

$$\begin{aligned}&\operatorname{curl} \mathcal{E}_2(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) + ik\overline{\chi}(\mathbf{y}) \left((\mathcal{E}_2(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) \right) \\ &\qquad = \operatorname{curl} \mathbf{M}_2(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) + ik\overline{\chi}(\mathbf{y}) \left((\mathbf{M}_2(\mathbf{y}) \times \hat{\boldsymbol{\nu}}(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y}) \right), \qquad \mathbf{y} \in \partial D. \end{aligned} \tag{2.12}$$

DEFINITION 2.1. *We say that the domain D and the impedance χ are a Herglotz pair if the unique solution $\mathcal{E}_1$ of the boundary value problem (2.7)–(2.9) and the unique solution $\mathcal{E}_2$ of the boundary value problem (2.10)–(2.12) can be represented by (2.2) for two suitable choices $\mathbf{g}_1, \mathbf{g}_2$ of the integral kernel $\mathbf{g}$. The vector fields $\mathbf{g}_1, \mathbf{g}_2$ that represent, respectively, $\mathcal{E}_1$, $\mathcal{E}_2$ are the generalized Herglotz kernels associated to the pair (D, χ).*

The class of the Herglotz pairs is not empty. In fact, it is easy to see by explicit computation that the class of the spheres of the center of the origin with constant boundary impedance belongs to it for every $\mathbf{a}, \mathbf{v} \in \mathbb{R}^3$ when the nonresonance condition on $-k^2$ is satisfied.

Let (D, χ) be a Herglotz pair $\forall k \in \Omega_2$, with respect to two given vectors $\mathbf{a}, \mathbf{v} \in \mathbb{R}^3$. Let $\mathbf{g}_1, \mathbf{g}_2$ be the corresponding Herglotz kernels; it is easy to prove that

$$\int_{\partial B} (\overline{\mathbf{g}}_1(\hat{\mathbf{x}}), \mathbf{E}_0(\hat{\mathbf{x}}, k, \boldsymbol{\alpha}, \boldsymbol{\omega}_{\boldsymbol{\alpha}}))\, d\lambda(\hat{\mathbf{x}}) = (\mathbf{v}, \boldsymbol{\omega}_{\boldsymbol{\alpha}}), \qquad \forall \boldsymbol{\alpha} \in \partial B, \quad \boldsymbol{\omega}_{\boldsymbol{\alpha}} \in \mathbb{R}^3, \tag{2.13}$$

$$\int_{\partial B} (\overline{\mathbf{g}}_2(\hat{\mathbf{x}}), \mathbf{E}_0(\hat{\mathbf{x}}, k, \boldsymbol{\alpha}, \boldsymbol{\omega}_{\boldsymbol{\alpha}}))\, d\lambda(\hat{\mathbf{x}}) = ik(\mathbf{v}, \boldsymbol{\omega}_{\boldsymbol{\alpha}})(\boldsymbol{\alpha}, \mathbf{a}), \qquad \forall \boldsymbol{\alpha} \in \partial B, \quad \boldsymbol{\omega}_{\boldsymbol{\alpha}} \in \mathbb{R}^3; \tag{2.14}$$

see [1,4].

Problem 1.1 will be solved in three steps:

(i) from the knowledge of the far field $\mathbf{E}_0(\hat{\mathbf{x}}, k, \boldsymbol{\alpha}, \boldsymbol{\omega}_{\boldsymbol{\alpha}})$ for several $\hat{\mathbf{x}}, \boldsymbol{\alpha}, \boldsymbol{\omega}_{\boldsymbol{\alpha}}$ determine, by (2.13) and (2.14), an approximation of the generalized Herglotz kernels $\mathbf{g}_1, \mathbf{g}_2$ of the pair (D, χ);
(ii) compute $\mathcal{E}_1$ and $\mathcal{E}_2$ from the knowledge of the vector fields $\mathbf{g}_1, \mathbf{g}_2$ using formula (2.2);
(iii) from the knowledge of the vector fields $\mathcal{E}_1$, $\mathcal{E}_2$, recover the shape ∂D and the impedance χ of the obstacle using relations (2.9),(2.12).

3. THE NUMERICAL METHOD

The numerical implementation of steps (i),(ii) is analogous to the one described in [4] and we omit it. Moreover, as in [4–6], step (iii) is reformulated as a global optimization problem.

Let (r, θ, φ) be the spherical coordinates of $\mathbb{R}^3$.

The implementation of step (iii) is done assuming that the obstacle is star-like and symmetric with respect to the x_3-axis; i.e., $\partial D = \{(r, \theta, \varphi) \mid r = f(\theta),\ 0 \le \theta \le \pi\}$, and the impedance is of the form $\chi(\theta) = \chi^{\mathrm{Re}}(\theta) + i\chi^{\mathrm{Im}}(\theta)$, $0 \le \theta \le \pi$. The smoothness of ∂D implies f smooth and the symmetry assumption made on ∂D implies $\frac{df}{d\theta}(0) = \frac{df}{d\theta}(\pi) = 0$.

Eliminating the impedance χ from the relations (2.9),(2.12), we obtain a new relation, that is

$$\begin{aligned} F_i(\mathbf{y}) \equiv{}& \left(\operatorname{curl}\, (\mathcal{E}_1(\mathbf{y}) - \mathbf{M}_1(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})\right)_i \left(((\mathcal{E}_2(\mathbf{y}) - \mathbf{M}_2(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})\right)_i \\ &- \left(\operatorname{curl}\, (\mathcal{E}_2(\mathbf{y}) - \mathbf{M}_2(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})\right)_i \left(((\mathcal{E}_1(\mathbf{y}) - \mathbf{M}_1(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})) \times \hat{\boldsymbol{\nu}}(\mathbf{y})\right)_i = 0, \\ & \mathbf{y} \in \partial D, \qquad i = 1, 2, 3, \end{aligned} \tag{3.1}$$

where $(\cdot)_i$ means the i^{th} component of the vector $\cdot$.

Condition (3.1) becomes a system of three complex equations involving f, $\frac{df}{d\theta}$, and it is particularly easy to solve for $\theta = \theta_1 = 0$ and $\theta = \theta_2 = \pi$, where it becomes a nonlinear system of equations in one unknown. In fact, we know that $\frac{df}{d\theta}(\theta_i) = 0$, $i = 1, 2$. We solve this overdetermined system in the least squares sense to obtain f_i^* approximate values of $f(\theta_i)$, $i = 1, 2$.

We approximate $f(\theta)$ with a truncated Fourier series, that is

$$f(\theta) \cong \tilde{f}(\theta) = a_0 + \sum_{l=1}^{L_r} \{a_l \cos l\theta + b_l \sin l\theta\}, \tag{3.2}$$

where L_r is a parameter that must be chosen depending on ∂D. For the simple obstacles considered later, we have chosen $L_r = 4$.

Let $\mathbf{c} = (a_0, a_1, b_1, \dots, a_{L_r}, b_{L_r})^\top$. Then the unknown boundary ∂D is obtained minimizing with respect to $\mathbf{c}$ the function

$$I(\mathbf{c}) = \left\{ \sum_{i=1}^{3} \int_0^{\pi} |F_i(\mathbf{y}(\theta))|^2 \, d\theta \right\}^{1/2} + \left\{ \sum_{i=1}^{3} p_i (\tilde{f}(\theta_i) - f_i^*)^2 \right\}^{1/2}, \tag{3.3}$$

where in the first term $\mathbf{y}(\theta) \equiv (r = \tilde{f}(\theta), \theta, 0)$, and the second one is a penalization term where $p_i \geq 0$, $i = 1, 2$ are weight factors. In the numerical experience shown in Section 4, we have chosen $p_1 = p_2 = 100$.

The minimization of $I(\mathbf{c})$ is done starting from the unit sphere ($a_0 = 1$, $a_l = b_l = 0$, $l = 1, \dots, L_r$) as an initial guess and using a quasi-Newton algorithm of the IMSL software library [7].

After recovering ∂D, that is f, we recover χ from (2.9),(2.12). In fact, given a positive integer N, we compute the estimates $\chi_n = \chi_n^{\mathrm{Re}} + i\chi_n^{\mathrm{Im}}$, $n = 0, \dots, N$ of the boundary impedance $\chi(\theta_n) = \chi^{\mathrm{Re}}(\theta_n) + i\chi^{\mathrm{Im}}(\theta_n)$, $\theta_n = n\frac{\pi}{N}$, $n = 0, \dots, N$ solving the linear system (2.9),(2.12) in the least squares sense for the unknowns χ_n, $n = 0, 1, \dots, N$.

4. NUMERICAL EXPERIENCE

The data used in our numerical experience, i.e., the far fields generated by the obstacles, are obtained solving numerically the boundary value problem (1.5)–(1.8) with the T-matrix method; see [8,9].

We have considered twenty-four different obstacles, that is all the possible combinations of the following six surfaces:

the sphere: $$r = 1, \qquad 0 \leq \theta \leq \pi, \tag{4.1}$$

the oblate ellipsoid: $$\left(\frac{2}{3}x_1\right)^2 + \left(\frac{2}{3}x_2\right)^2 + x_3^2 = 1, \tag{4.2}$$

the prolate ellipsoid: $$x_1^2 + x_2^2 + \left(\frac{2}{3}x_3\right)^2 = 1, \tag{4.3}$$

the short cylinder: $$\left(\left(\frac{2}{3}x_1\right)^2 + \left(\frac{2}{3}x_2\right)^2\right)^5 + x_3^{10} = 1, \tag{4.4}$$

the pseudo apollo: $$r = \frac{3}{5}\left(\frac{17}{4} + 2\cos 3\theta\right)^{1/2}, \qquad 0 \leq \theta \leq \pi, \tag{4.5}$$

the Vogel's peanut: $$r = \frac{3}{2}\left(1 - \frac{3}{4}\sin^2\theta\right)^{1/2}, \qquad 0 \leq \theta \leq \pi, \tag{4.6}$$

and the following four impedances:

$$\chi_1(\theta) = 1 - i, \qquad 0 \leq \theta \leq \pi, \tag{4.7}$$

$$\chi_2(\theta) = \left(1 - \frac{1}{2}i\right)(1 + \sin\theta), \qquad 0 \leq \theta \leq \pi, \tag{4.8}$$

$$\chi_3(\theta) = \begin{cases} 1, & 0 \leq \theta < \dfrac{\pi}{2}, \\ i, & \dfrac{\pi}{2} \leq \theta \leq \pi, \end{cases} \tag{4.9}$$

$$\chi_4(\theta) = \begin{cases} 1, & 0 \leq \theta < \dfrac{\pi}{3}, \\ i, & \dfrac{\pi}{3} \leq \theta < \dfrac{2}{3}\pi, \\ 1, & \dfrac{2}{3}\pi \leq \theta \leq \pi. \end{cases} \tag{4.10}$$

We note the characteristic length L of the obstacles considered is one; then, the resonance region is $k = O(1)$.

Table 4.1. The numerical results.

Reconstruction	Object D	Impedance χ	$E_{L_2}^{\partial D}$	$E_{L_2}^{\chi}$
1	Sphere	χ_1	0	0
2	"	χ_2	0.0003	0.017
3	"	χ_3	0.0168	0.158
4	"	χ_4	0.0446	0.2206
5	Oblate ellipsoid	χ_1	0.0068	0.0287
6	"	χ_2	0.0051	0.0413
7	"	χ_3	0.0098	0.1321
8	"	χ_4	0.0437	0.2424
9	Prolate ellipsoid	χ_1	0.0066	0.0214
10	"	χ_2	0.007	0.0346
11	"	χ_3	0.0545	0.1804
12	"	χ_4	0.0633	0.2296
13	Short cylinder	χ_1	0.0577	0.2233
14	"	χ_2	0.0574	0.3035
15	"	χ_3	0.0996	0.3170
16	"	χ_4	0.0466	0.2529
17	Pseudo apollo	χ_1	0.0591	0.2822
18	"	χ_2	0.068	0.2615
19	"	χ_3	0.1857	0.3266
20	"	χ_4	0.1303	0.3554
21	Vogel's peanut	χ_1	0.0435	0.2475
22	"	χ_2	0.0278	0.2883
23	"	χ_3	0.1293	0.2555
24	"	χ_4	0.0488	0.2856

The numerical results given in Table 4.1 are obtained using the following parameters:

$$\Omega_1 = \left\{ \hat{\mathbf{x}}(\theta_l, \phi_j) \mid \theta_l = \frac{\pi}{12} l,\ l = 1, \dots, 11;\ \phi_j = \frac{\pi}{11} j,\ j = 0, \dots, 10 \right\} \cup \{0, 0\} \cup \{\pi, 0\},$$

$$\Omega_2 = \Bigg\{ (\hat{\boldsymbol{\alpha}}_{lj}, \boldsymbol{\omega}_{ljm}) \mid \boldsymbol{\alpha}_{lj} = \hat{\mathbf{x}}(\theta_l, \phi_j), \boldsymbol{\omega}_{ljm} = (-\sin\gamma_m \sin\phi_j + \cos\gamma_m \cos\theta_l \cos\phi_j,$$
$$\sin\gamma_m \cos\phi_j + \cos\gamma_m \cos\theta_l \sin\phi_j, -\cos\gamma_m \sin\theta_l)^{\mathsf{T}} : \theta_l = \frac{\pi}{12} l,\ l = 1, \dots, 11;$$
$$\phi_j = \frac{2\pi}{9} j,\ j = 0, \dots, 8;\ \gamma_m = \frac{\pi}{2} m,\ m = 0, 1 \Bigg\},$$

$$\Omega_3 = \{1, 9, 25\},$$
$$\mathbf{v} \in \{(1,0,0), (0,1,0), (0,0,1)\},$$
$$\mathbf{a} \in \{(1,0,0), (0,1,0), (0,0,1)\}.$$

Moreover, the expansions in vector spherical harmonics of $\mathbf{E}_0, \mathbf{g}_1, \mathbf{g}_2$ are truncated at $L_{\max} = 9$; that is, we use the first $L_{\max} + 1 = 10$ l-channels corresponding to $2L_{\max}(L_{\max} + 2) = 198$ terms.

Finally, the last two columns in Table 4.1 report the performance indices of the reconstructions; that is

$$E_{L_2}^{\partial D} = \left\{ \frac{\sum_{n=0}^{N} [\tilde{f}(\theta_n) - f(\theta_n)]^2}{\sum_{n=0}^{N} f(\theta_n)^2} \right\}^{1/2}, \tag{4.11}$$

and

$$E_{L_2}^{\chi} = \left\{ \frac{\sum_{n=0}^{N} \left| \chi_n^{\mathrm{Re}} + i\chi_n^{\mathrm{Im}} - \chi(\theta_n) \right|^2}{\sum_{n=0}^{N} |\chi(\theta_n)|^2} \right\}^{1/2}. \tag{4.12}$$

We remark that the reconstruction of ∂D is consistently of higher quality than the reconstruction of χ.

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